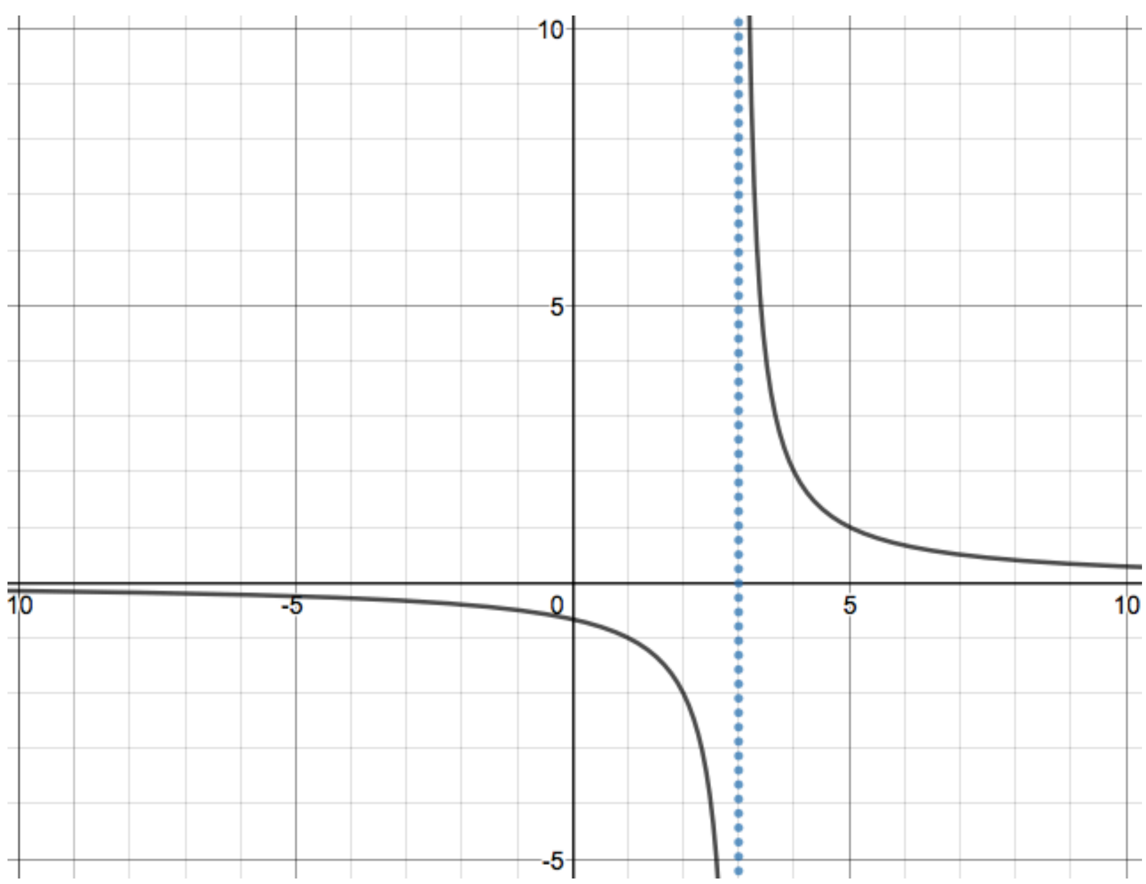
1.4 Infinite Limits

Objectives:

- Determine One-Sided Infinite Limits
- Find and Sketch Vertical Asymptotes of the graph of a function
- Determine Limits at Infinity
- Determine the horizontal asymptotes of a function
- Determine Infinite limits at infinity

Lets examine the following graph of the function $f(x) = \dfrac{2}{x-3}$



An **Infinite Limit** is a limit where $f(x)$ increases or decreases without bound as $x \to c$.

$$\lim_{x \to c^-} f(x) = \pm\infty \quad \text{and/or} \quad \lim_{x \to c^+} f(x) = \pm\infty$$

** The limit does not actually exist, however you will still hear the phrase “The limit is (+/-) infinity”

Extension 1

In the explanation above you saw one case. There are three other general cases. Sketch all four cases then explain the limits. (Hint: Think about what the graph may look like.)

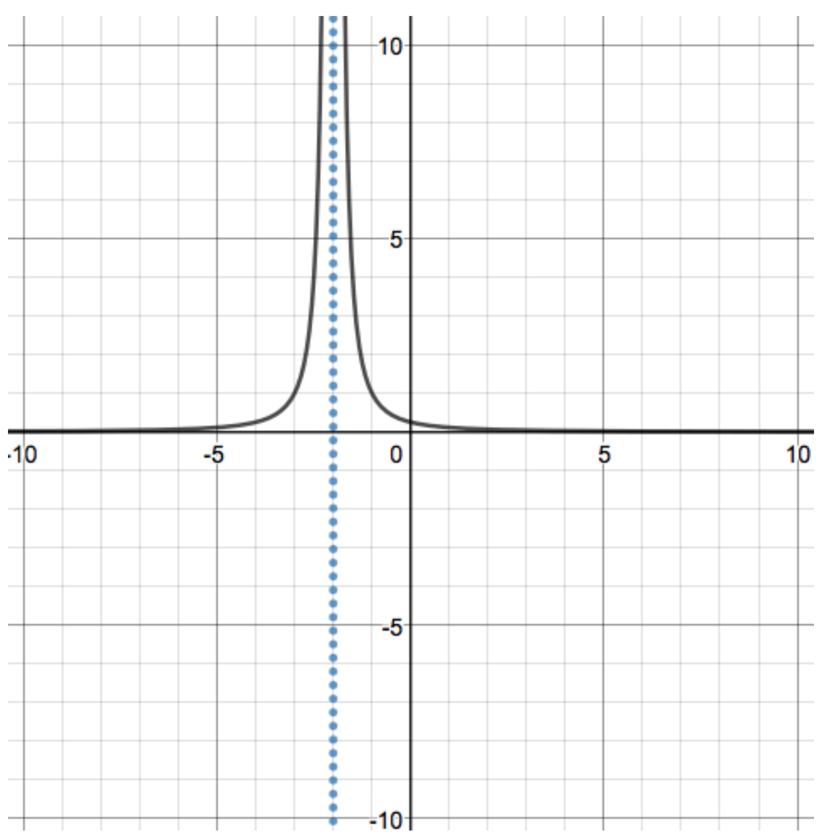
Extension 2

Now that you have sketched all cases, use Theorem: Limit Existence from the previous section to develop a conjecture about limits involving infinity.

Example 1

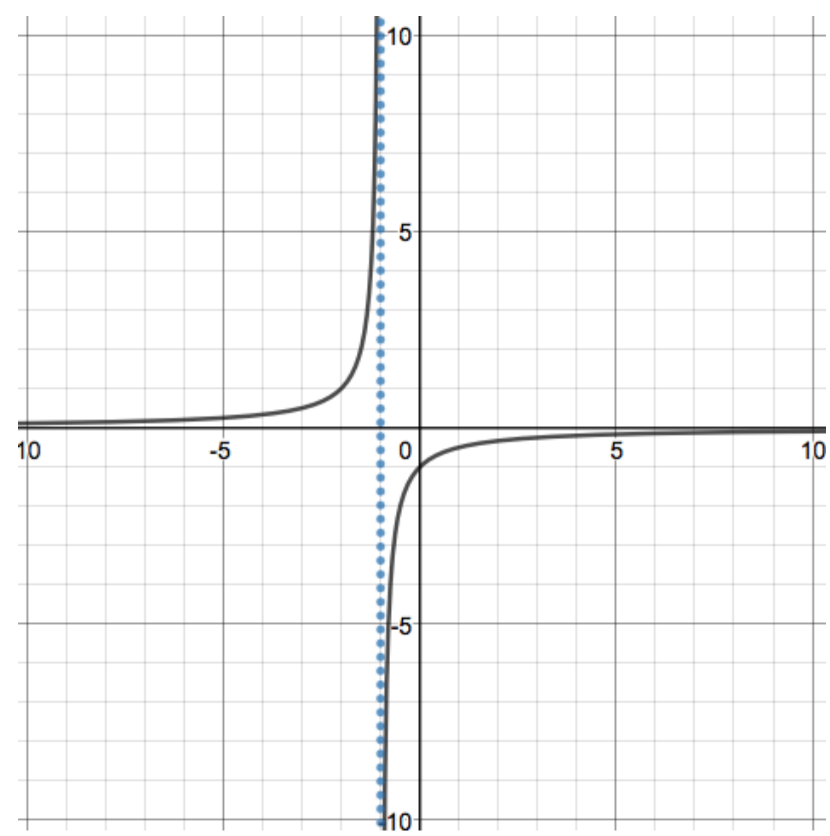
Determine the limit of the given function as x approaches -1 from the left and from the right.

a.



$$f(x) = \frac{1}{(x+1)^2}$$

b.



$$f(x) = \frac{-1}{x+1}$$

https://youtu.be/6K3-f0wU-7c

The blue dotted line in the above examples are Vertical Asymptotes. You should recall Rational Function from Pre-Calculus and how to find the following:

- x and y-intercepts
- Vertical Asymptotes
- Holes
- Horizontal Asymptotes
- Slant or Other Asymptotes

However, in Calculus we define a vertical asymptote differently…

Definition: Vertical Asymptotes

If $f(x) \to \pm\infty$ as $x \to c^-$ or $x \to c^+$, then the vertical line $x = c$ is a vertical asymptote of the function f.

Example 2

Find the following one-sided limits:

1. $\lim_{x \to -1^-} \dfrac{x^2 + 3x}{x + 1}$

2. $\lim_{x \to -1^+} \dfrac{x^2 + 3x}{x + 1}$

https://youtu.be/lIA7g-A8-zI

Properties of Infinite Limits

Let c and L be real numbers and f and g be functions where

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L \text{ , then}$$

1. Sum or Difference: $\lim_{x \to c} [f(x) \pm g(x)] = \infty$

2. Product: $\lim_{x \to c} [f(x) \cdot g(x)] = \infty \text{ , } if\ L > 0$ and $\lim_{x \to c} [f(x) \cdot g(x)] = -\infty \text{ , } if\ L < 0$

3. Quotient: $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$

**These properties can be extended to one-sided limits **

Extension 3

Read each of the Properties of Infinite Limits. In your own words, explain how each one makes sense to you. Use mathematical reasoning to support your explanation as much as possible.

Example 3

Find the limit: $\lim_{x \to 0} \left(2 + \frac{3}{x^3}\right)$

https://youtu.be/bnYmv610ZfM

Example 4

Find the limit: $\lim_{x \to \frac{\pi}{2}^+} \frac{x+2}{\tan x}$

https://youtu.be/kvnddyFrFng

Example 5

Find the limit: $\lim_{x \to \pi^-} -2 \csc x$

https://youtu.be/0o_J3IM9F58

Limits at Infinity.

What happens if $x \to \infty$? In Pre-Calculus, the *end behavior* of a function was discussed. Especially that of a rational function, $r(x) = \frac{f(x)}{g(x)}$.

From Pre-Calculus, the horizontal asymptotes describe the end behavior of a funtion:

1. If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$
2. If the degree of the numerator is equal to that of the denominator the the equation of the horizontal asymptote is $y =$ *ratio of the leading coefficients*.
3. If the degree of the numerator is greater than the degree of the denominator then a different type of asymptote exists and is the quotient of the numerator divided by the denominator.

In Calculus, however, Horizontal Asymptotes are defined this way,

Definition of Horizontal Asymptote

The line $y = L$ is a horizontal asymptote of the function f when

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \qquad \lim_{x \to \infty} f(x) = L$$

The interpretation of this is: as x continues on to infinity, the y value the function approaches is L.

Limits at Infinity

Let r be a positive rational number and c any real number, then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0$$

Furthermore if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{c}{x^r} = 0$$

Example 6

Find the limit: $\lim_{x \to \infty} \left(3 - \frac{5}{x^2}\right)$

https://youtu.be/vxiFAmr4N-g

Example 7

Find the limit: $\lim_{x \to \infty} \left(\frac{2x - 3}{x - 1}\right)$

https://youtu.be/_J_8ybfmoss

Example 8

Find each limit and discuss your findings.

1. $\lim_{x \to \infty} \frac{3x - 4}{2x^2 - 1}$
2. $\lim_{x \to \infty} \frac{3x^2 - 4}{2x^2 - 1}$
3. $\lim_{x \to \infty} \frac{3x^3 - 4}{2x^2 - 1}$

https://youtu.be/_Yzmh4OzheU

Generalities for finding limits at Infinities

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0.

2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist.

Extension 4

Compare the "Generalities for finding limits at Infinities" to the precalculus horizontal asymptotes. What similarities and/or differences do you recognize?

Example 9

Find the limits:

1. $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

2. $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

https://youtu.be/GcSsI28eEXI

Example 10

Find the limit: $\lim_{x\to\infty} \sin x$

https://youtu.be/63MIv-JSB3k

Example 11

Find each limit.

1. $\lim_{x\to\infty} x^5$

2. $\lim_{x\to-\infty} x^5$

https://youtu.be/hkQVkkf_J9Q

Example 12

Find each limit.

1. $\lim_{x\to\infty} \frac{3x^2-6x}{x+1}$

2. $\lim_{x\to-\infty} \frac{3x^2-6x}{x+1}$

https://youtu.be/swUazJ5AhkY