1.2 Evaluating Limits Analytically

Objectives:

- Evaluating a limit using properties of limits
- Develop strategies for finding limits
- Evaluate a limit using equivalent forms and rationalizing techniques

Theorem: Basic Limits

For the following, let b, c be real numbers and n a positive integer

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Example 1

Evaluate $\lim_{x \to 2} 3$

Example 2

Find $\lim_{x \to -4} x$

Example 3

Find $\lim_{x \to 2} x^2$

Video for Examples 1, 2, &3: https://youtu.be/wUP_USmdx20

Direct Substitution will work so long as substituting in c does not cause an indeterminate form, i.e. division by zero, infinity over infinity, etc. You will learn techniques for these types later in the lesson.

Theorem: Properties of Limits

For the following, let b, c be real numbers and n a positive integer. Let f and g be well defined functions with the following limits:

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M$$

1. Scalar Multiplication: $\lim_{x \to c} b[f(x)] = b \cdot \lim_{x \to c} f(x) = b \cdot L$
2. Sum or Difference: $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = L \pm M$
3. Product: $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = L \cdot M$
4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{L}{M}, \quad M \neq 0$
5. Power: $\lim_{x \to c} f(x)^n = \left(\lim_{x \to c} f(x) \right)^n = L^n$

Example 4

Evaluate $\lim_{x \to 2} (4x^2 + 3)$ using the properties of limits.

Worked example: https://youtu.be/wwKPmbPBdU0

Theorem: Limits of Polynomials and Rational Functions

1. Let p be a polynomial function and c be a real number, then.
   $\lim_{x \to c} p(x) = p(c)$
2. Let r be a rational function $r(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomial functions and c is a real number, then
   $\lim_{x \to c} r(x) = r(c) = \dfrac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} = \dfrac{p(c)}{q(c)}$

Example 5

Find the limit: $\lim_{x \to 1} \dfrac{x^2 + 3}{x + 1}$

Worked example: https://youtu.be/zgv3beFxk-Y

Extension: 1

Let *n* be a positive integer. Use the Power Property of Limits to derive a formula for the following limit:

$$\lim_{x \to c} \sqrt[n]{x}$$

Is there anything to consider when *n* is odd? Even? If so, explain.

Limit of a Composite Function

Let *f* and *g* be function where $\lim_{x \to c} g(x) = M$ and $\lim_{x \to M} f(x) = f(M)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(M)$$

Example 6

Use the Limit of a Composite Function to find: $\lim_{x \to 0} \sqrt{x^3 + 4}$

Worked example: https://youtu.be/7jjcAwXpI2s

Example 7

Use the Limit of a Composite Function to find: $\lim_{x \to 3} \sqrt[3]{2x^2 + 9}$

Worked example: https://youtu.be/vvFrWhKsHRw

Extension 2

Determine if you can use Direct Substitution in the two above examples. Why or why does it work? Explain.

Theorem: Limits of Trigonometric Functions

Let *c* be a real number in the domain of each trigonometric function

1. $\lim_{x \to c} \cos x = \cos c$
2. $\lim_{x \to c} \sin x = \sin c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \sec x = \sec c$
5. $\lim_{x \to c} \csc x = \csc c$
6. $\lim_{x \to c} \cot x = \cot c$

Example 8

Evaluate: $\lim_{x \to \frac{\pi}{2}} \tan x$

Example 9

Evaluate: $\lim_{x \to \frac{3\pi}{4}} x \sin x$

Example 10

Evaluate: $\lim_{x \to \frac{\pi}{3}} \cos^2 x$

Worked example for 8, 9 & 10: https://youtu.be/oMLGr7Hx9g0

Strategies and Methods for Finding Limits

Consider the functions:

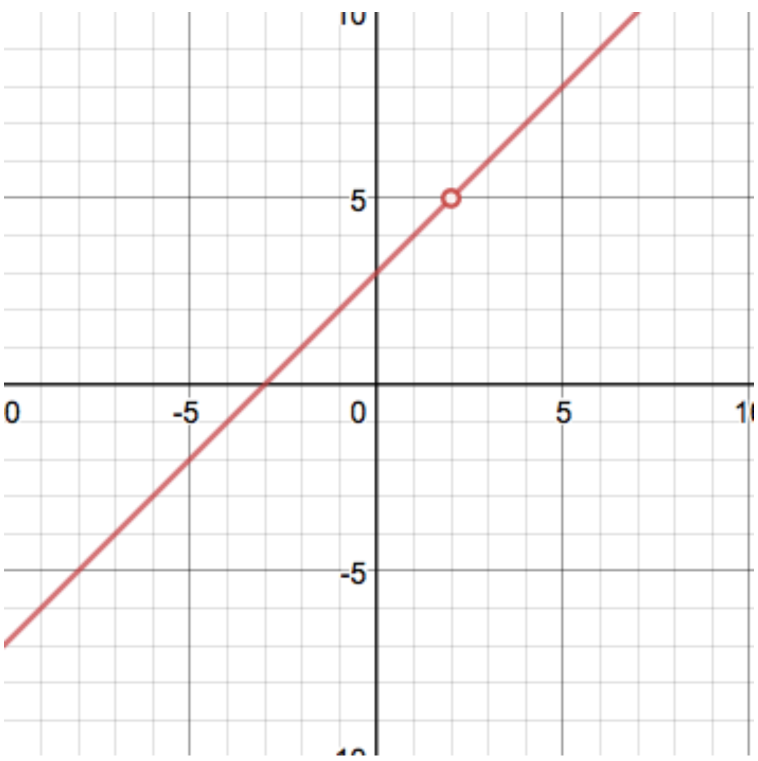
$$f(x) = \frac{x^2 + 2x - 3}{x - 1} \qquad \text{and} \qquad g(x) = x + 3$$

$f(x)$ has an x-intercept at x = -3, y-intercept at y = 3 and hole at x = 1.

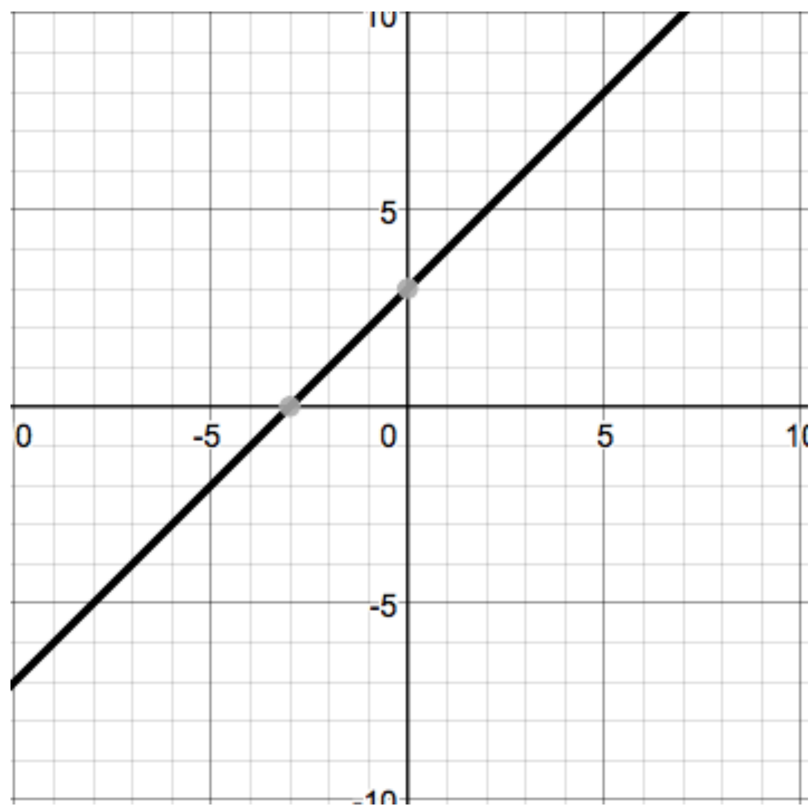
$g(x)$ has an x-intercept at x = -3, y-intercept at y = 3.

Graphing these two functions, we see they coincide at all points save one, when x = 1.

$f(x)$



$g(x)$



Theorem: Equivalent Functions Except at One Point

Let *c* be a real number and $f(x) = g(x)$ for all $x \neq c$. If $\lim_{x \to c} g(x)$ exists, then $\lim_{x \to c} f(x)$ also exists, and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

Suggestion for finding Limits

1. Check direct substitution first. Make sure that no indeterminate form occurs.
2. If an indeterminate form exists, try to simplify the given equation into an equivalent form, then use direct substitution.
3. If the equation has a radical, try using rationalization to simplify to an equivalent form, then use direct substitution.

Example 11

Find the limit: $\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3}$

Worked example: https://youtu.be/SghxEDPQ-AQ

Example 12

Find the limit: $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$

Worked example: https://youtu.be/A6BYBDzWXp8

Two Important Trigonometric Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Example 13

Find the limit: $\lim_{x \to 0} \frac{\tan x}{x}$

Worked example: https://youtu.be/f1cNre3O0yY

Example 14

Find the limit: $\lim_{x \to 0} \frac{\sin 4x}{x}$

Worked example: https://youtu.be/RVGvN2JNf4M