Chapter 4 Polynomials

4.1 Polynomial Functions

Objectives:
- Define a polynomial
- Divide polynomials: Long Division and Synthetic
- Apply the Remainder Theorem, The Factor Theorem, and the connections between remainders and factors.
- Determine the maximum number of zeros of a polynomial.

A **polynomial** is an algebraic expression that can be written in the form:

$$p(x) = a_n x^n + a_{n-1} x^{x-1} + \ldots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

Where $n$is a non-negative integer, x is a variable, and each a_n represents a coefficient. A **coefficient** is the number usually in front of a scalar.variable When the coefficient is without a variable, as in the case of a_0, it is named the **constant**. The characteristics of a polynomial are:

- All exponents are whole numbers
- No variable is contained in the denominator.
- No variable is under a radical.

A polynomial that consists of only a constant term is called a **constant polynomial**. The **zero polynomial** is a constant polynomial 0.

The **degree** of a polynomial is the highest power of x that appears with a non-zero coefficient. The coefficient that precedes this x with the highest power is known as the **leading coefficient**.

The Division Algorithm and Parts of a Division

The division algorithm defines division, that is:

Dividend = Divisor · Quotient + Remainder

- Divisor = what your are dividing by
- Dividend = what you are dividing into
- Quotient = the result when dividing a divisor into a dividend
- Remainder = anything that's "left over" from a division. Only exists if the division is not evenly.

Polynomial Long Division

Just like “old school” division.

Example 1

Divide $3x^4 - 8x^2 - 11x + 1$ by $x - 2$

https://youtu.be/La6GShniaTw

Synthetic Division

Synthetic division is a fast way to divide a polynomial when the divisor is linear, i.e $x + c$ or $x - c$. This type of division can **only** be used with linear divisors.

Example 2

Divide using synthetic division: $3x^4 - 8x^2 - 11x + 1$ by $x - 2$

https://youtu.be/z3Ls0u1-DnA

Example 3

Divide using synthetic division: $x^5 + 5x^4 + 6x^3 - x^2 + 4x + 29 \div x + 3$

https://youtu.be/HNkwbXad6nI

Division Algorithm for Polynomials

If a polynomial $f(x)$ is divided by a nonzero polynomial $d(x)$, divisor, then there is a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$ such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

Where $r(x) = 0$ or has a degree less than the divisor $d(x)$.

The Division algorithm can be used to determine if a divisor is a factor of a dividend. A **factor** is a divisor that when divided into a dividend has a remainder equal to zero. The division algorithm in this case would be

$$f(x) = d(x) \cdot q(x)$$

Example 4

Determine if $2x^2 + 1$ is a factor of $6x^3 - 4x^2 + 3x - 2$

https://youtu.be/y9jDLIWekMk

Remainder

When polynomial is divided by a linear polynomial such as $x - c$ or $x + c$ the remainder is always a constant. That is if $f(x) = x^3 - 2x^2 - 4x + 5$ is divided by $x - 3$ the quotient is $x^2 + x - 1$ and the remainder is 2.

We can check using the Division Algorithm

$$\begin{aligned} x^3 - 2x^2 - 4x + 5 &= (x-3)(x^2+x-1) + 2 \\ &= x^3 - 2x^2 - 4x + 3 + 2 \\ &= x^3 - 2x^2 - 4x + 5 \end{aligned}$$

Now check $f(3)$.

$$\begin{aligned} f(3) &= (3)^3 - 2(3)^2 - 4(3) + 5 \\ &= 27 - 18 - 12 + 5 \\ &= 32 - 30 \\ &= 2 \end{aligned}$$

Notice that the remainder from dividing by $x - c$ is the same value as $f(c)$.

Remainder Theorem

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$

Example 5

Find the remainder when $x^{79} + 3x^{24} + 5$ is divided by $x - 1$

https://youtu.be/p5kraEb7130

Example 6

Find the remainder when $3x^4 - 8x^2 + 11x + 1$ is divided by $x + 2$

https://youtu.be/dVAE96sLwj8

Recall that a divisor is called a factor when the remainder equals zero. We can use this understanding and the Remainder Theorem to name another theorem…

Factor Theorem

A polynomial $f(x)$has a linear factor $x - c$ if and only if $f(c) = 0$

Example 7

A. Show that $x-3$ is a factor of x^3-4x^2+2x+3 using the Factor Theorem.

B. Find $q(x)$ such that $x^3-4x^2+2x+3=(x-3)\cdot q(x)$

https://youtu.be/icllkk8WedQ

When we studied quadratics we studied the connection between the x-intercepts, factors, the zeros of a function f, and the solutions of $f(x)=0$. Here we see these connections in regards to polynomials.

Zeros, X-intercepts, Solutions, and Factors

Let $f(x)$ be polynomial function. IF r is a real number that satisfies any of the following statements, then all are true about r.

- r is a zero of the function f.
- r is an x-intercept of the graph of the function f.
- $x=r$ is a solution/root of the equation $f(x)=0$.
- $x-r$ is a factor of the polynomial $f(x)$.

This states that a zero, an x-intercept, a solution, and the value of r in a linear factor of the form $x-r$ are all the same for a polynomial. In addition, the x-intercept correspond to the linear factors of $f(x)$of the form $x-r$.

Example 8

For $f(x)=6x^3+7x^2-x-2$, find the following:

A. The x-intercepts of the graph of f.

B. The zeros of f

C. The solutions of $f(x)=0$

D. The linear factors of $f(x)$

https://youtu.be/NEfNjGjQRhg

If a polynomial $f(x)$ has four zeros a, b, c, and d, then $f(x)=(x-a)(x-b)(x-c)(x-d)$ and $f(x)$ has a degree of 4.

Example 9

Find three (3) different polynomial of different degrees that have 1, 2, 3 and -5 as zeros.

https://youtu.be/CmWNIoezIOQ

Correlation between Degree and Number of Zeros

A polynomial of degree n has at most n distinct real zeros.

Example 10

Determine the maximum number of real zeros of the following polynomial:

$$p(x) = 18x^4 - 51x^3 - 111x^2 - 56 + 1$$

https://youtu.be/feFVJJQFgj8