6.4 Square Matrices

Objectives

- Define an $n \times n$ identity matrix
- Find the inverse of an invertible matrix
- Solve square systems of equations using inverse matrices

A system of equations that has equal amounts of equations as it does variables are called **square systems**. This section will explore a method of solving square systems that does not involve row reduction (Guass-Jordan Elimination). However, this method can only be used if the system has a unique solution. It does not work for inconsistent systems or ones that have infinitely many solutions.

Instead of writing a single augmented matrix to represent a system, we will break up the system into 3 matrices that represent the coefficients, the variables, and the constants. Consider the following matrix:

$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y &= 5 \\ x + 2y + z &= -1 \end{aligned}$$

We can use three matrices to represent each part like so,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad B = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

Coefficient Matrix   Variable Matrix   Constant

Example 1

Use the three matrices above to show that $AX = B$ is equivalent to the system of equations given in the introduction.

Inverse Matrices and The Identity Matrix

To solve equations you have been using inverse operations to “undo” and isolate variables. For example, in order to solve the following equation $3x = 12$, what would you do? Would you “divide” both sides by 3? Or multiply both sides by the multiplicative inverse of $\frac{1}{3}$? Either way, you are making use of inverse operations in order to solve the equation. A matrix equation can be solved in a similar fashion making use of a multiplicative inverse. To understand this method we must first consider the **Identity Matrix,** I_n with dimensions $n \times n$.

The Identity Matrix

The $n \times n$ **Identity Matrix** I_n is the matrix with n row and n columns that contains 1's on the main diagonal, that is the diagonal from the top left to the bottom right and 0's in all other entries. The identity matrix has the following property that for any matrix A with dimensions $n \times n$:

$$A \cdot I_n = I_n \cdot A = A$$

Example 2

For matrix C, verify that $C \cdot I_2 = I_2 \cdot C = C$

$$C = \begin{pmatrix} 3 & 2 \\ 5 & 7 \end{pmatrix}$$

Inverse Matrices

An $n \times n$ A is called **invertible** or **nonsingular** if there exists an $n \times n$ matrix B such that $A \cdot B = I_n$. The matrix B is called the inverse of A and is written A^{-1} and

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n$$

NOTE: Not all matrices are invertible

Example 3

Verify that matrix B is the inverse of A.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$$

Methods for Finding the Inverse of a 2×2 Matrix.

Example 4 Using $A^{-1} = \begin{pmatrix} x & u \\ y & v \end{pmatrix}$

a. Find the inverse of matrix $A = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$

b. Verify inverses.

Example 5 Using the Glass-Jordan Elimination

Find the inverse of matrix $A = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$

A formula for A^{-1} of a 2×2 Matrix

if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 6 Use the formula.

Find the inverse of matrix $A = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$

Method for Finding the Inverse of a $n \times n$ Matrix.

Example 7 Guass-Jordan Elimination

a. Find the inverse of matrix A
b. Verify inverses.

$$\text{Let } A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

Solving Square Systems Using A^{-1}

Now that inverse matrices have been discussed, recall how to solve an equation $A \cdot X = B$. Now $A, X,$ and B represent matrices so we can solve the matrix equation in the following manner:

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ I_nX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

where the unique solution of the matrix equation is $X = A^{-1} \cdot B$

If A is not invertible then the system is either inconsistent or has infinitely many solutions.

Example 8
Use an inverse matrix to solve:

$$\begin{aligned} x + y &= 2 \\ 2x - 9y &= 15 \end{aligned}$$

Example 9
Use an inverse matrix to solve:

$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y \quad &= 5 \\ x + 2y + z &= -1 \end{aligned}$$

Curve Fitting

Just as two points determine a unique line, three non-collinear points determine a unique parabola. Moreover, a polynomial of degree n can be determined bu $n + 1$ non-collinear points. We can use matrix methods to find equations of curves that pass through a set of points.

Example 10
Find the equation of the parabola that passes through the points (1, -4), (-1, 10) and (4, -25)