4.5 Fundamental Theorem of Algebra

Section 1: Factor Theorem

In Exercises 1–6, determine if $g(x)$ is a factor of $f(x)$ without using synthetic or long division.

1. $f(x) = x^{10} + x^8 \qquad g(x) = x - 1$
2. $f(x) = x^6 - 10 \qquad g(x) = x - 2$
3. $f(x) = 3x^4 - 6x^3 + 2x - 1 \qquad g(x) = x + 1$
4. $f(x) = x^5 - 3x^2 + 2x - 1 \qquad g(x) = x - 2$
5. $f(x) = x^3 - 2x^2 + 5x - 4 \qquad g(x) = x + 2$
6. $f(x) = 10x^{75} - 8x^{65} + 6x^{45} + 4x^{32} - 2x^{15} + 5$
   $g(x) = x - 1$

1. $f(x) = x^{10} + x^8 \qquad g(x) = x - 1$
   No
2. $f(x) = x^6 - 10 \qquad g(x) = x - 2$
   No
3. $f(x) = 3x^4 - 6x^3 + 2x - 1 \qquad g(x) = x + 1$
   No
4. $f(x) = x^5 - 3x^2 + 2x - 1 \qquad g(x) = x - 2$
   No
5. $f(x) = x^3 - 2x^2 + 5x - 4 \qquad g(x) = x + 2$
   No
6. $f(x) = 10x^{75} - 8x^{65} + 6x^{45} + 4x^{32} - 2x^{15} + 5$
   $g(x) = x - 1$
   No

Section 2: Multiplicity

In Exercises 7–10, list the zeros of the polynomial and state the multiplicity of each zero.

7. $f(x) = x^{54}\left(x + \frac{4}{5}\right)$
8. $g(x) = 3\left(x + \frac{1}{6}\right)\left(x - \frac{1}{5}\right)\left(x + \frac{1}{4}\right)$
9. $h(x) = 2x^{15}(x - \pi)^{14}[x - (\pi + 1)]^{13}$
10. $k(x) = (x - \sqrt{7})^7(x - \sqrt{5})^5(2x - 1)$

7. $x = 0$ (multiplicity 54);
   $x = -\frac{4}{5}$ (multiplicity 1)
8. $x = -\frac{1}{6}$ (multiplicity 1);
   $x = \frac{1}{5}$ (multiplicity 1);
   $x = -\frac{1}{4}$ (multiplicity 1)
9. $x = 0$ (multiplicity 15);
   $x = \pi$ (multiplicity 14);
   $x = \pi + 1$ (multiplicity 13)
10. $x = \sqrt{7}$ (multiplicity 7);
    $x = \sqrt{5}$ (multiplicity 5);
    $x = -\frac{1}{2}$ (multiplicity 1)

Section 3 Writing polynomials as the product of linear factors

In Exercises 11–22, find all the zeros of f in the complex number system; then write $f(x)$ as a product of linear factors.

11. $f(x) = x^2 - 2x + 5$
12. $f(x) = x^2 - 4x + 13$
13. $f(x) = 3x^2 + 2x + 7$
14. $f(x) = 3x^2 - 5x + 2$
15. $f(x) = x^3 - 27$ *Hint:* Factor first.
16. $f(x) = x^3 + 125$
17. $f(x) = x^3 + 8$
18. $f(x) = x^6 - 64$
    Hint: Let $u = x^3$ and factor $u^2 - 64$ first.
19. $f(x) = x^4 - 1$
20. $f(x) = x^4 - x^2 - 6$
21. $f(x) = x^4 - 3x^2 - 10$
22. $f(x) = 2x^4 - 7x^2 - 4$

11. $x = 1 \pm 2i$
    $f(x) = (x - 1 - 2i)(x - 1 + 2i)$
12. $x = 2 \pm 3i$
    $f(x) = (x - 2 + 3i)(x - 2 - 3i)$
13. $x = -\frac{1}{3} \pm \frac{2\sqrt{5}}{3}i$
    $f(x) = \left(x + \frac{1}{3} - \frac{2\sqrt{5}}{3}i\right) \cdot \left(x + \frac{1}{3} + \frac{2\sqrt{5}}{3}i\right)$
14. $x = 1$ or $\frac{2}{3}$; $f(x) = (3x - 2)(x - 1)$
15. $x = 3$ or $-\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$
    $f(x) = (x - 3)\left(x + \frac{3}{2} - \frac{3\sqrt{3}}{2}i\right) \cdot \left(x + \frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)$
16. $x = -5$ or $\frac{5}{2} \pm \frac{5\sqrt{3}}{2}i$
    $f(x) = (x + 5)\left(x - \frac{5}{2} - \frac{5\sqrt{3}}{2}i\right)^2 \cdot \left(x - \frac{5}{2} + \frac{5\sqrt{3}}{2}i\right)$

Section 4: Writing a polynomial with given zeros

In Exercises 23–44, find a polynomial $f(x)$ with real coefficients that satisfies the given conditions. Some of the problems have many correct answers.

23. degree 3; only zeros are $1, 7, -4$

24. degree 3; only zeros are 1 and -1

25. degree 6; only zeros are $1, 2, \pi$

26. degree 5; only zero is 2

27. degree 3; zeros $-3, 0, 4$; $f(5) = 80$

28. degree 3; zeros $-1, \frac{1}{2}, 2$; $f(0) = 2$

29. zeros include $2 + i$ and $2 - i$

30. zeros include $1 + 3i$ and $1 - 3i$

31. zeros include 2 and $2 + i$

32. zeros include 3 and $4i - 1$

33. zeros include $-3, 1 - i, 1 + 2i$

34. zeros include $1, 2 + i, 3i - 1$

35. degree 2; zeros $1 + 2i$ and $1 - 2i$

36. degree 4; zeros $3i$ and $-3i$, each of multiplicity 2

37. degree 4; only zeros are $4, 3 + i$, and $3 - i$

38. degree 5; zeros 2 of multiplicity 3, i, and $-i$

39. degree 6; zeros 0 of multiplicity 3 and 3, $1 + i$, $1 - i$, each of multiplicity 1

40. degree 6; zeros include i of multiplicity 2 and 3

41. degree 2; zeros include $1 + i$; $f(0) = 6$

42. degree 2; zeros include $3 + i$; $f(2) = 3$

43. degree 3; zeros include i and 1; $f(-1) = 8$

44. degree 3; zeros include $2 + 3i$ and -2; $f(2) = -3$

17. $x = -2$ or $1 \pm i\sqrt{3}$
$f(x) = (x + 2)(x - 1 - i\sqrt{3})(x - 1 + i\sqrt{3})$

18. $x = -1 \pm i\sqrt{3}$ or $1 \pm i\sqrt{3}$ or ± 2;
$f(x) = (x - 2)(x + 2)(x - 1 - i\sqrt{3}) \cdot (x - 1 + i\sqrt{3})(x + 1 - i\sqrt{3})(x + 1 + i\sqrt{3})$

19. $x = \pm 1$ or $\pm i$;
$f(x) = (x - 1)(x - i)(x + 1)(x + i)$

20. $x = \pm\sqrt{3}$ or $\pm i\sqrt{2}$;
$f(x) = (x - \sqrt{3})(x + \sqrt{3})(x - i\sqrt{2})(x + i\sqrt{2})$

21. $x = \pm\sqrt{5}$ or $\pm i\sqrt{2}$
$f(x) = (x - \sqrt{5})(x + \sqrt{5})(x - i\sqrt{2}) \cdot (x + i\sqrt{2})$

22. $x = \pm 2$ or $\pm\frac{1}{\sqrt{2}}i$
$f(x) = 2\left(x - \frac{1}{\sqrt{2}}i\right)\left(x + \frac{1}{\sqrt{2}}i\right)(x - 2)(x + 2)$

23. degree 3; only zeros are $1, 7, -4$
Possible answer: $f(x) = (x - 1)(x - 7)(x + 4)$

24. degree 3; only zeros are 1 and -1
Possible answer: $f(x) = (x - 1)^2(x + 1)$

25. degree 6; only zeros are $1, 2, \pi$
Possible answer: $f(x) = (x - 1)(x - 2)^2(x - \pi)^3$

26. degree 5; only zero is 2
Possible answer: $f(x) = (x - 2)^5$

27. degree 3; zeros $-3, 0, 4$; $f(5) = 80$
$f(x) = 2x(x - 4)(x + 3)$

28. degree 3; zeros $-1, \frac{1}{2}, 2$; $f(0) = 2$

29. zeros include $2 + i$ and $2 - i$
$f(x) = x^2 - 4x + 5$

28. $f(x) = 2(x + 1)\left(x - \frac{1}{2}\right)(x - 2)$

30. zeros include $1 + 3i$ and $1 - 3i$
$f(x) = x^2 - 2x + 10$

31. zeros include 2 and $2 + i$
$f(x) = (x - 2)(x^2 - 4x + 5)$

32. zeros include 3 and $4i - 1$
$f(x) = (x - 3)(x^2 + 2x + 17)$

33. zeros include $-3, 1 - i, 1 + 2i$
$f(x) = (x + 3)(x^2 - 2x + 2)(x^2 - 2x + 5)$

34. zeros include $1, 2 + i, 3i - 1$
$f(x) = (x - 1)(x^2 - 4x + 5)(x^2 + 2x + 10)$

35. degree 2; zeros $1 + 2i$ and $1 - 2i$
$f(x) = x^2 - 2x + 5$

36. degree 4; zeros $3i$ and $-3i$, each of multiplicity 2
$f(x) = (x^2 + 9)^2$

37. degree 4; only zeros are $4, 3 + i$, and $3 - i$
$f(x) = (x - 4)^2(x^2 - 6x + 10)$

38. degree 5; zeros 2 of multiplicity 3, i, and $-i$
$f(x) = (x - 2)^3(x^2 + 1)$

39. degree 6; zeros 0 of multiplicity 3 and 3, $1 + i$, $1 - i$, each of multiplicity 1
$f(x) = x^3(x - 3)(x^2 - 2x + 2)$

40. degree 6; zeros include i of multiplicity 2 and 3
$f(x) = (x^2 + 1)^2(x - 3)^2$

41. degree 2; zeros include $1 + i$; $f(0) = 6$
$f(x) = 3(x^2 - 2x + 2)$

42. degree 2; zeros include $3 + i$; $f(2) = 3$
$f(x) = \frac{3}{2}x^2 - 9x + 15$

43. degree 3; zeros include i and 1; $f(-1) = 8$
$f(x) = -2x^3 + 2x^2 - 2x + 2$

44. degree 3; zeros include $2 + 3i$ and -2; $f(2) = -3$

44. $f(x) = -\frac{1}{12}x^3 + \frac{1}{6}x^2 - \frac{5}{12}x - \frac{13}{6}$

Section 5: Find a polynomial given complex zeros.

In Exercises 45–48, find a polynomial with complex coefficients that satisfies the given conditions.

45. degree 2; zeros i and $1 - 2i$

46. degree 2; zeros $2i$ and $1 + i$

47. degree 3; zeros 3, i, and $2 - i$

48. degree 4; zeros $\sqrt{2}, -\sqrt{2}, 1 + i$, and $1 - i$

Possible answers:

45. degree 2; zeros i and $1 - 2i$
$f(x) = x^2 - (1 - i)x + (2 + i)$

46. degree 2; zeros $2i$ and $1 + i$
$f(x) = x^2 - (1 + 3i)x + (-2 + 2i)$

47. degree 3; zeros 3, i, and $2 - i$
$f(x) = x^3 - 5x^2 + (7 + 2i)x - (3 + 6i)$

48. degree 4; zeros $\sqrt{2}, -\sqrt{2}, 1 + i$, and $1 - i$
$f(x) = x^4 - 2x^3 + 4x - 4$

Section 6: Find all zeros (real and complex).

In Exercises 49–56, one zero of the polynomial is given; find all the zeros.

49. $x^3 - 2x^2 - 2x - 3$; zero 3

50. $x^3 + x^2 + x + 1$; zero i

51. $x^4 + 3x^3 + 3x^2 + 3x + 2$; zero i

52. $x^4 - x^3 - 5x^2 - x - 6$; zero i

53. $x^4 - 2x^3 + 5x^2 - 8x + 4$; zero 1 of multiplicity 2

54. $x^4 - 6x^3 + 29x^2 - 76x + 68$; zero 2 of multiplicity 2

55. $x^4 - 4x^3 + 6x^2 - 4x + 5$; zero $2 - i$

56. $x^4 - 5x^3 + 10x^2 - 20x + 24$; zero $2i$

49. $x^3 - 2x^2 - 2x - 3$; zero 3 $3, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

50. $x^3 + x^2 + x + 1$; zero i
$-1, \pm i$

51. $x^4 + 3x^3 + 3x^2 + 3x + 2$; zero i
$-1, -2, \pm i$

52. $x^4 - x^3 - 5x^2 - x - 6$; zero i
$3, -2, \pm i$

53. $x^4 - 2x^3 + 5x^2 - 8x + 4$; zero 1 of multiplicity 2
$1, \pm 2i$

54. $x^4 - 6x^3 + 29x^2 - 76x + 68$; zero 2 of multiplicity 2
$2, 1 \pm 4i$

55. $x^4 - 4x^3 + 6x^2 - 4x + 5$; zero $2 - i$
$\pm i, 2 \pm i$

56. $x^4 - 5x^3 + 10x^2 - 20x + 24$; zero $2i$
$3, 2, \pm 2i$