4.5 Fundamental Theorem of Algebra

Section 1: Factor Theorem

In Exercises 1–6, determine if $g(x)$ is a factor of $f(x)$ without using synthetic or long division.

1. $f(x) = x^{10} + x^8 \qquad g(x) = x - 1$
2. $f(x) = x^6 - 10 \qquad g(x) = x - 2$
3. $f(x) = 3x^4 - 6x^3 + 2x - 1 \qquad g(x) = x + 1$
4. $f(x) = x^5 - 3x^2 + 2x - 1 \qquad g(x) = x - 2$
5. $f(x) = x^3 - 2x^2 + 5x - 4 \qquad g(x) = x + 2$
6. $f(x) = 10x^{75} - 8x^{65} + 6x^{45} + 4x^{32} - 2x^{15} + 5$
   $g(x) = x - 1$

Section 2: Multiplicity

In Exercises 7–10, list the zeros of the polynomial and state the multiplicity of each zero.

7. $f(x) = x^{54}\left(x + \frac{4}{5}\right)$
8. $g(x) = 3\left(x + \frac{1}{6}\right)\left(x - \frac{1}{5}\right)\left(x + \frac{1}{4}\right)$
9. $h(x) = 2x^{15}(x - \pi)^{14}[x - (\pi + 1)]^{13}$
10. $k(x) = (x - \sqrt{7})^7(x - \sqrt{5})^5(2x - 1)$

Section 3 Writing polynomials as the product of linear factors

In Exercises 11–22, find all the zeros of f in the complex number system; then write $f(x)$ as a product of linear factors.

11. $f(x) = x^2 - 2x + 5$
12. $f(x) = x^2 - 4x + 13$
13. $f(x) = 3x^2 + 2x + 7$
14. $f(x) = 3x^2 - 5x + 2$
15. $f(x) = x^3 - 27$ *Hint:* Factor first.
16. $f(x) = x^3 + 125$
17. $f(x) = x^3 + 8$
18. $f(x) = x^6 - 64$
    Hint: Let $u = x^3$ and factor $u^2 - 64$ first.
19. $f(x) = x^4 - 1$
20. $f(x) = x^4 - x^2 - 6$
21. $f(x) = x^4 - 3x^2 - 10$
22. $f(x) = 2x^4 - 7x^2 - 4$

Section 4: Writing a polynomial with given zeros

In Exercises 23–44, find a polynomial $f(x)$ with real coefficients that satisfies the given conditions. Some of the problems have many correct answers.

23. degree 3; only zeros are $1, 7, -4$

24. degree 3; only zeros are 1 and -1

25. degree 6; only zeros are $1, 2, \pi$

26. degree 5; only zero is 2

27. degree 3; zeros $-3, 0, 4$; $f(5) = 80$

28. degree 3; zeros $-1, \frac{1}{2}, 2$; $f(0) = 2$

29. zeros include $2 + i$ and $2 - i$

30. zeros include $1 + 3i$ and $1 - 3i$

31. zeros include 2 and $2 + i$

32. zeros include 3 and $4i - 1$

33. zeros include $-3, 1 - i, 1 + 2i$

34. zeros include $1, 2 + i, 3i - 1$

35. degree 2; zeros $1 + 2i$ and $1 - 2i$

36. degree 4; zeros $3i$ and $-3i$, each of multiplicity 2

37. degree 4; only zeros are $4, 3 + i$, and $3 - i$

38. degree 5; zeros 2 of multiplicity 3, i, and $-i$

39. degree 6; zeros 0 of multiplicity 3 and 3, $1 + i$, $1 - i$, each of multiplicity 1

40. degree 6; zeros include i of multiplicity 2 and 3

41. degree 2; zeros include $1 + i$; $f(0) = 6$

42. degree 2; zeros include $3 + i$; $f(2) = 3$

43. degree 3; zeros include i and 1; $f(-1) = 8$

44. degree 3; zeros include $2 + 3i$ and -2; $f(2) = -3$

Section 5: Find a polynomial given complex zeros.

In Exercises 45–48, find a polynomial with complex coefficients that satisfies the given conditions.

45. degree 2; zeros i and $1 - 2i$

46. degree 2; zeros $2i$ and $1 + i$

47. degree 3; zeros 3, i, and $2 - i$

48. degree 4; zeros $\sqrt{2}, -\sqrt{2}, 1 + i$, and $1 - i$

Section 6: Find all zeros (real and complex).

In Exercises 49–56, one zero of the polynomial is given; find all the zeros.

49. $x^3 - 2x^2 - 2x - 3$; zero 3

50. $x^3 + x^2 + x + 1$; zero i

51. $x^4 + 3x^3 + 3x^2 + 3x + 2$; zero i

52. $x^4 - x^3 - 5x^2 - x - 6$; zero i

53. $x^4 - 2x^3 + 5x^2 - 8x + 4$; zero 1 of multiplicity 2

54. $x^4 - 6x^3 + 29x^2 - 76x + 68$; zero 2 of multiplicity 2

55. $x^4 - 4x^3 + 6x^2 - 4x + 5$; zero $2 - i$

56. $x^4 - 5x^3 + 10x^2 - 20x + 24$; zero $2i$